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# The analysis of 2 × 2 factorial fracture experiments with brittle materials

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Experiments conducted at all combinations of the levels of two or more factors are called factorial experiments. Factorial experiments have been shown to be more efficient in exploring the effects of external factors on a response variable than non-factorial arrangements of factor levels. This paper presents a methodology for the analysis of test results obtained at all combinations of the levels of two two-level factors. The response variable is assumed to follow the two parameter Weibull distribution with a shape parameter that, although unknown, does not vary with the factor levels. The scale parameter on the other hand may vary in various ways with the levels of the factors. These assumptions well reflect the behavior of the fracture strength of glassy polymers as it is influenced by factors such as water sorption and silanation of filler. The purpose of the analysis is (1) to compute interval estimates of the common shape parameter and (2) to assess whether either factor singly or in combination with the other affects the Weibull scale parameter. The procedure is illustrated with an example using simulated data for which only one of the factors had a real effect. A further example is given using shear strength measurements made in a $2^2$ factorial experiment conducted on a glassy polymer material used in dental restorations. 

## 1. Introduction

The two-parameter Weibull distribution was first used to model the random variation of fracture strength of brittle materials by Weibull himself [1]. Subsequently it has been extensively used to model the fracture strength of ceramics, polymers and glasses. Much of this work suggests that while external factors may have an effect on the Weibull scale parameter they generally have little or no effect on the shape parameter. In the present paper we examine the combined effect of two factors on fracture strength. The factors can be qualitative such as the presence or absence of some ingredient, or whether the specimen was or was not immersed in fluid. They may be quantitative such as the ambient temperature or humidity.

Factorial experiments have an advantage in efficiency over experiments in which arbitrary combinations of levels are used [2]. Analysis methods for factorial tests with exponential response were developed in [3, 4]. It was shown in [3] that the methodology was not robust if the data were actually from a Weibull population. This paper presents methodology for analyzing such experiments to determine whether neither, one, or both factors have a significant influence on a Weibull distributed random variable such as fracture strength. It is an adaptation to the context of fracture tests of results presented in [5, 6] and applied to rolling contact fatigue in [7]. A primary purpose of this paper is to introduce to the community of materials scientists a new, statistically valid and powerful methodology for the design and analysis of fracture experiments.

## 2. The model

The statement that the fracture strength of a population of items operating under a specified set of conditions follows the two parameter Weibull model implies that the probability that the strength of a randomly selected item does not exceed a value '$x$' is expressible as:

$$P[\text{strength} < x] = F(x) = 1 - \exp\left[-\left(\frac{x}{\eta}\right)^{\beta}\right] \quad (1)$$

$\beta$ is known as the shape parameter and $\eta$ the scale parameter or characteristic strength. Both are positive.

A useful property of Weibull distributed random variables is that when they are multiplicatively transformed, i.e., by $y = cx$, the distribution of the transformed variable $y$ remains Weibull with the same shape parameter but with a scale parameter of $c\eta$.

In applying the Weibull model to fracture experiments we assume that $\beta$ does not vary with the level of external factors. As noted above this is consistent with assuming that the external factors serve to increase or decrease the scale parameter $\eta$ by a multiplicative amount. Specifically, if factor $A$ has "$a$" levels and factor $B$ has "$b$" levels, the strength distribution at the conditions corresponding to level $i$ of factor $A$ and level $j$ of factor $B$ is:

$$F(x) = 1 - \exp\left[ -\left( \frac{x}{\eta_{ij}} \right)^{\beta} \right] \tag{2}$$

One may think of the levels of factor $A$ as forming the rows and the levels of factor $B$ the columns of a two-way table or layout.

The scale parameter $\eta_{ij}$ may now be expressed in terms of a multiplicative row effect $a_i$ due to the $i$th level of factor $A$, a column effect $b_j$ due to the $j$th level of factor $B$ and, an interaction effect $c_{ij}$ due to the particular synergy of row $i$ and column $j$, i.e.,

$$\eta_{ij} = a_i b_j c_{ij} \eta; \quad (i = 1, \dots, a,\ j = 1, \dots, b) \tag{3}$$

$\eta$ is a base level scale parameter value. Introducing the additional constraints:

$$\prod_{i=1}^{a} a_i = 1 \tag{4}$$

$$\prod_{j=1}^{b} b_j = 1 \tag{5}$$

and

$$\prod_{i=1}^{a} c_{ij} = \prod_{j=1}^{b} c_{ij} = 1 \tag{6}$$

serves to define $\eta$ as the geometric mean of the cell scale parameter values taken over all the cells; i.e.,

$$\eta = \left[ \prod_{j=1}^{b} \prod_{i=1}^{a} \eta_{ij} \right]^{1/ab} \tag{7}$$

Thus, for example, given the following $2 \times 2$ table of scale parameter values:

| | |
|---|---|
| $\eta_{11} = 2$ | $\eta_{12} = 4$ |
| $\eta_{21} = 3$ | $\eta_{22} = 6$ |

We have,

$$\eta = (2 \times 4 \times 3 \times 6)^{1/4} = \sqrt{12}$$

From the constraints:

$$a_2 = \frac{1}{a_1}$$

$$b_2 = \frac{1}{b_1}$$

$$c_{11} = \frac{1}{c_{12}} = c_{22} = \frac{1}{c_{21}}$$

Equating the numerical values of $\eta_{ij}$ to their multiplicative expressions gives:

$$\eta_{11} = a_1 b_1 c_{11} \sqrt{12} = 2$$

$$\eta_{12} = \frac{a_1}{b_1} \cdot \frac{1}{c_{11}} \sqrt{12} = 4$$

$$\eta_{21} = \frac{b_1}{a_1 c_{11}} \cdot \sqrt{12} = 3$$

$$\eta_{22} = \frac{c_{11}}{a_1 b_1} \sqrt{12} = 6$$

The solutions are:

$$a_1 = \sqrt{2/3}, \quad a_2 = \sqrt{3/2}$$

$$b_1 = \sqrt{1/2}, \quad b_2 = \sqrt{2}$$

$$c_{11} = c_{12} = c_{21} = c_{22} = 1.0$$

In this instance multiplicative row and column factors sufficed to account for all the $\eta_{ij}$ values. The cell specific $c_{ij}$ values were all unity. For the values in the table above it may be said that "interaction" is absent.

In general if $c_{ij}$ is unity for all $i$ and $j$, we say that interaction does not occur. When interaction is absent the data may be 'explained' by the simpler 'reduced' model wherein:

$$\eta_{ij} = a_i b_j \eta; \quad (i = 1, \dots, a,\ j = 1, \dots, b) \tag{8}$$

Similarly if, in addition, factor $B$ has no effect, $\eta_{ij}$ may be written:

$$\eta_{ij} = \eta_i = a_i\, \eta; \quad (i = 1, \dots, a) \tag{9}$$

While if factor $A$ has no effect:

$$\eta_{ij} = \eta_j = b_j \eta; \quad (j = 1, \dots, b) \tag{10}$$

Finally, if there are no row, column or interaction effects the model simply reduces to:

$$\eta_{ij} = \eta \tag{11}$$

## 3. Fracture tests

A fracture test is presumed to be conducted for each combination of factor levels. For simplicity we take the sample size $n$ to be the same for each cell in the $a \times b$ array. The $n$ items are presumed to be tested until the $r$th ($r \leq n$) smallest fracture strength in the sample occurs and that the remaining $n - r$ specimens do not fracture. Generally $r = n$ in fracture tests. Life tests on the other hand are often censored ($r < n$). The total sample size is thus $abn$ and the number of failed specimens is $abr$.

TABLE I ML estimation equations for factorial experiments under various models

| Model | ML shape parameter found by solving | Equation for $\hat{\eta}_{ij}^{\hat{\beta}}$ |
|---|---|---|
| $\eta_{ij} = a_i b_j c_{ij} \eta$ | $\frac{1}{\hat{\beta}_1} + \frac{s_{..}}{abr} - \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{T_{ij}}{v_{ij}} = 0$ | $v_{ij}/r$ |
| $\eta_{ij} = a_i b_i \eta$ $(a = b = 2)$ | $\frac{1}{\hat{\beta}_2} + \frac{s_{..}}{abr} - \frac{1}{k}\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{T_{ij}}{v_{i.}v_{.j}} = 0$ $\left(k \equiv \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{v_{ij}}{v_{i.}v_{.j}}\right)$ | $kv_{i.}v_{.j}/abr$ |
| $\eta_{ij} = a_i \eta$ | $\frac{1}{\hat{\beta}_3} + \frac{s_{..}}{abr} - a^{-1}\sum_{i=1}^{a}\left(\frac{\sum_{j=1}^{b} T_{ij}}{\sum_{j=1}^{b} v_{ij}}\right) = 0$ | $\sum_{j=1}^{b} v_{ij}/br$ |
| $\eta_{ij} = b_j \eta$ | $\frac{1}{\hat{\beta}_4} + \frac{s_{..}}{abr} - b^{-1}\sum_{j=1}^{b}\left(\frac{\sum_{i=1}^{a} T_{ij}}{\sum_{i=1}^{a} v_{ij}}\right) = 0$ | $\sum_{i=1}^{a} v_{ij}/ar$ |
| $\eta_{ij} = \eta$ | $\frac{1}{\hat{\beta}_5} + \frac{s_{..}}{abr} - \sum_{i=1}^{a}\sum_{j=1}^{b} T_{ij} \Big/ \sum_{i=1}^{a}\sum_{j=1}^{b} v_{ij} = 0$ | $\sum_{i=1}^{a}\sum_{j=1}^{b} v_{ij}/abr$ |

When the observed strengths within each cell are sorted from low to high the $k$th ordered value is denoted $X_{ij(k)}$.

## 4. Estimation

In [5] the method of maximum likelihood is applied to estimate the shape parameter and effects under each of the four models described by Equations 3 and 8–11. The corresponding shape parameter estimates are denoted $\hat{\beta}_1$ to $\hat{\beta}_5$ respectively. For model number 2, $\hat{\beta}_2$ and the estimates of the effects $a_i$ and $b_j$ must generally be found by the simultaneous solution of $a + b + 1$ nonlinear equations. However, for the special case of the $2 \times 2$ layout ($a = b = 2$) $\hat{\beta}_2$ may be solved separately and then the effect estimates computed. In what follows we restrict consideration to the case where $a = b = 2$. The estimates of the scale parameters $\eta_{ij}$ are obtained by multiplying the relevant effect estimates and the estimate of the base level scale parameter.

Table I lists the equations for estimating $\hat{\beta}_1$ to $\hat{\beta}_5$. The auxiliary quantities used in this table are defined as follows:

$$v_{ij} = \sum_{1}^{n} X_{ij(k)}^{\hat{\beta}} \tag{12}$$

$$v_{i.} = \left\{\prod_{j=1}^{b} v_{ij}\right\}^{1/b} \tag{13}$$

$$v_{.j} = \left\{\prod_{i=1}^{a} v_{ij}\right\}^{1/a} \tag{14}$$

$$v_{..} = \left\{\prod_{i=1}^{a}\prod_{j=1}^{b} v_{ij}\right\}^{1/ab} \tag{15}$$

$$T_{ij} = \sum_{k=1}^{n} X_{ij(k)}^{\hat{\beta}} \ln X_{ij(k)} \tag{16}$$

$$s_{ij} = \sum_{k=1}^{r} \ln X_{ij(k)} \tag{17}$$

$$s_{..} = \sum_{i=1}^{a}\sum_{j=1}^{b} s_{ij} \tag{18}$$

Also listed in Table I is the expression for the maximum likelihood estimate of the cell scale parameters raised to a power equal to the shape parameter estimate appropriate to that model. Thus, for example, to estimate $\eta_{ij}$ under the last model of Table I, one computes:

$$\hat{\eta}_{ij} = \left[\sum_{i=1}^{a}\sum_{j=1}^{b} v_{ij}/abr\right]^{1/\hat{\beta}_5} \tag{19}$$

Except for model 2, the shape parameter estimates obtained under the other models are special cases of the shape parameter estimate applicable to $k$ groups of Weibull data under the assumption that the shape parameter is the same for all groups. This estimate was first discussed in [8] for the case $k = 2$. It was generalized in [9] and elaborated upon in the context of the analysis of rolling contact fatigue experiments in [10].

$\hat{\beta}_1$, for example, is the estimate of $\beta$ obtained when each of the cells in the data array is taken as a group. In this case $k = ab$. $\hat{\beta}_3$ results when each *row* is taken as a group, i.e., the columns are collapsed. In this case $k = a$. Correspondingly, $\hat{\beta}_4$ is obtained by collapsing rows and treating the data in each *column* as a group. This gives $k = b$. $\hat{\beta}_5$ results when all of the data in the array are treated as a single large group, i.e., $k = 1.0$.

## 5. Test for appropriate model

The full model given by Equation 3 is the least restrictive. Under this model $\eta_{ij}$ is estimated using only the data in the cell $(i,\ j)$ along with the common shape parameter estimate $\hat{\beta}_1$ obtained using all of the data in the entire array. Succeeding models are successively more restrictive with the last model $\eta_{ij} = \eta$ representing the case where the entire sample of $abn = 4n$ items come from a single Weibull population. When a restrictive model is inappropriate, a consequence is that the shape parameter estimated under that model will tend to be smaller than it is when an appropriate model is used.

It is shown in [5] that one may use the ratio of shape parameter estimates as the basis for a test of whether a more restrictive model is tenable. For example, if interaction is absent, $\hat{\beta}_1/\hat{\beta}_2$ should be about unity. If interaction is present however, $\hat{\beta}_2$ will be relatively smaller than $\hat{\beta}_1$ and the ratio $\hat{\beta}_1/\hat{\beta}_2$ will therefore tend to be larger than unity.

In the language of hypothesis testing, our so called null hypothesis would be that interaction is absent, i.e.,

$$H_0: c_{ij} = 1.0 \quad (\text{all } i, j)$$

This hypothesis would be rejected with significance level $\alpha = 0.05$ if

$$\hat{\beta}_1/\hat{\beta}_2 > (\hat{\beta}_1/\hat{\beta}_2)_{1-\alpha} = (\hat{\beta}_1/\hat{\beta}_2)_{0.95} \tag{20}$$

$(\hat{\beta}_1/\hat{\beta}_2)_{0.95}$ represents the upper 95th percentile of the distribution of the ratio $(\hat{\beta}_1/\hat{\beta}_2)$ applicable when the hypothesis is true. It serves as a measure of the relative rarity of larger $\hat{\beta}_1/\hat{\beta}_2$ ratios. Only 5% of the time will a larger value be encountered due to chance alone when interaction is absent. If we encounter a larger value we proclaim that interaction exists and tolerate a 5% risk that our proclamation is wrong. If the hypothesis of no interaction were accepted one could proceed to test $\hat{\beta}_1/\hat{\beta}_3$ to see if the column effects are also negligible.

## 6. Monte Carlo results

Monte Carlo simulation was used to produce 10,000 simulated $2 \times 2$ factorial experiments in which the data in all cells were drawn from a common Weibull population. This was done for values of sample size $n$ and censoring number $r$ ranging from $n = r = 2$ to $n = r = 10$. For each simulated experiment the values $\hat{\beta}_1 - \hat{\beta}_5$ were computed using the equations in Table I. The ratios of $\hat{\beta}_1$ to the other values of $\hat{\beta}$ were calculated for each experiment and sorted from low to high to determine the percentiles.

Table II lists the upper 90, 95 and 99% points of these ratios along with the 5, 10, 50, 90 and 95% points of

$$v \equiv \hat{\beta}_1/\beta \tag{21}$$

These latter values are used for setting confidence limits on the shape parameter $\beta$ as in [9]. For 90% confidence limits one uses:

$$\hat{\beta}/v_{0.95} < \beta < \hat{\beta}/v_{0.05} \tag{22}$$

A median unbiased estimate of the common shape parameter may be computed as:

$$\hat{\beta}'' = \hat{\beta}/v_{0.50} \tag{23}$$

## 7. Discriminating power of the tests for row, column and interaction effects

Additional Monte Carlo studies were performed to evaluate the ability of the methodology discussed above to detect real effects. It is shown in [5] that in the absence of interaction effects the probability of detecting a row effect of magnitude $a_1$ (with $a_2 = 1/a_1$) increases with the magnitude of $a_1^{\beta}$. Simulations were run with $n = r = 3$ and $n = r = 10$ using various choices of $a_1$ and setting $\eta_{11} = \eta_{12} = a_1$ and $\eta_{21} = \eta_{22} = 1/a_1$. The probability of failing to detect an effect of magnitude $a_1$ using a significance level $\alpha = 0.10$ is shown in Fig. 1 as a function of $a_1^{\beta}$. Fig. 2 is a comparable plot giving the probability of failing to detect an interaction effect of magnitude $c_{11}$ as a function of $c_{11}^{\beta}$. It is seen that an interaction effect is more likely to be detected than a row (or column) effect of comparable size. These figures show that $2 \times 2$ factorial tests with $n$ (and $r$) as small as 10 are quite sensitive for brittle materials wherein the shape parameter $\beta$ is typically large. For example with $\beta = 10$, a multiplicative factor as small as $a_1 = 1.04$ will be detected with a probability of about 65%. ($1.04^{10} = 1.5$).

## 8. Numerical examples

To illustrate the analysis on data which are known to conform to its inherent assumptions, a $2 \times 2$ array was developed using simulation. Two uncensored samples of size $n = 5$ were generated from a Weibull population

TABLE II Selected percentage points of the distributions $\hat{\beta}_1/\beta$ and $\hat{\beta}_1/\hat{\beta}_k$ $(k = 2-5)$

| | | $\hat{\beta}_1/\beta$ | | | | | $\hat{\beta}_1/\hat{\beta}_2$ | | | $\hat{\beta}_1/\hat{\beta}_3$ and $\hat{\beta}_1/\hat{\beta}_4$ | | | $\hat{\beta}_1/\hat{\beta}_5$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $r$ | 0.05 | 0.10 | 0.50 | 0.90 | 0.95 | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 |
| 2 | 2 | 0.944 | 1.069 | 1.764 | 3.284 | 4.019 | 1.742 | 2.060 | 3.160 | 2.066 | 2.439 | 3.729 | 2.417 | 2.889 | 4.538 |
| 3 | 3 | 0.870 | 0.960 | 1.357 | 2.068 | 2.362 | 1.307 | 1.437 | 1.762 | 1.437 | 1.599 | 1.942 | 1.601 | 1.782 | 2.220 |
| 4 | 2 | 0.861 | 0.979 | 1.610 | 3.015 | 3.690 | 1.741 | 2.063 | 3.132 | 2.078 | 2.440 | 3.701 | 2.420 | 2.879 | 4.412 |
| 4 | 4 | 0.857 | 0.929 | 1.238 | 1.705 | 1.903 | 1.184 | 1.261 | 1.461 | 1.285 | 1.365 | 1.576 | 1.378 | 1.481 | 1.714 |
| 5 | 3 | 0.755 | 0.832 | 1.181 | 1.801 | 2.049 | 1.290 | 1.417 | 1.726 | 1.438 | 1.575 | 1.912 | 1.575 | 1.745 | 2.169 |
| 5 | 5 | 0.859 | 0.920 | 1.179 | 1.549 | 1.683 | 1.131 | 1.186 | 1.305 | 1.209 | 1.270 | 1.417 | 1.277 | 1.346 | 1.522 |
| 6 | 3 | 0.735 | 0.810 | 1.151 | 1.755 | 1.997 | 1.288 | 1.413 | 1.722 | 1.437 | 1.574 | 1.906 | 1.568 | 1.740 | 2.165 |
| 6 | 6 | 0.857 | 0.913 | 1.140 | 1.455 | 1.560 | 1.100 | 1.141 | 1.236 | 1.161 | 1.208 | 1.322 | 1.214 | 1.267 | 1.396 |
| 7 | 3 | 0.722 | 0.796 | 1.131 | 1.727 | 1.967 | 1.288 | 1.412 | 1.721 | 1.435 | 1.571 | 1.904 | 1.566 | 1.737 | 2.153 |
| 7 | 7 | 0.862 | 0.912 | 1.114 | 1.392 | 1.489 | 1.082 | 1.120 | 1.212 | 1.134 | 1.176 | 1.272 | 1.180 | 1.227 | 1.339 |
| 8 | 4 | 0.689 | 0.745 | 0.997 | 1.380 | 1.538 | 1.171 | 1.235 | 1.421 | 1.263 | 1.339 | 1.529 | 1.348 | 1.441 | 1.655 |
| 8 | 8 | 0.867 | 0.911 | 1.099 | 1.345 | 1.433 | 1.071 | 1.099 | 1.164 | 1.112 | 1.144 | 1.219 | 1.149 | 1.189 | 1.273 |
| 9 | 4 | 0.678 | 0.733 | 0.980 | 1.359 | 1.515 | 1.170 | 1.235 | 1.418 | 1.262 | 1.337 | 1.527 | 1.347 | 1.440 | 1.655 |
| 9 | 9 | 0.872 | 0.911 | 1.088 | 1.317 | 1.391 | 1.061 | 1.089 | 1.149 | 1.099 | 1.130 | 1.199 | 1.131 | 1.165 | 1.243 |
| 10 | 5 | 0.663 | 0.710 | 0.914 | 1.207 | 1.318 | 1.118 | 1.163 | 1.272 | 1.188 | 1.242 | 1.371 | 1.251 | 1.313 | 1.462 |
| 10 | 10 | 0.873 | 0.913 | 1.076 | 1.285 | 1.353 | 1.053 | 1.073 | 1.125 | 1.085 | 1.112 | 1.169 | 1.115 | 1.144 | 1.215 |

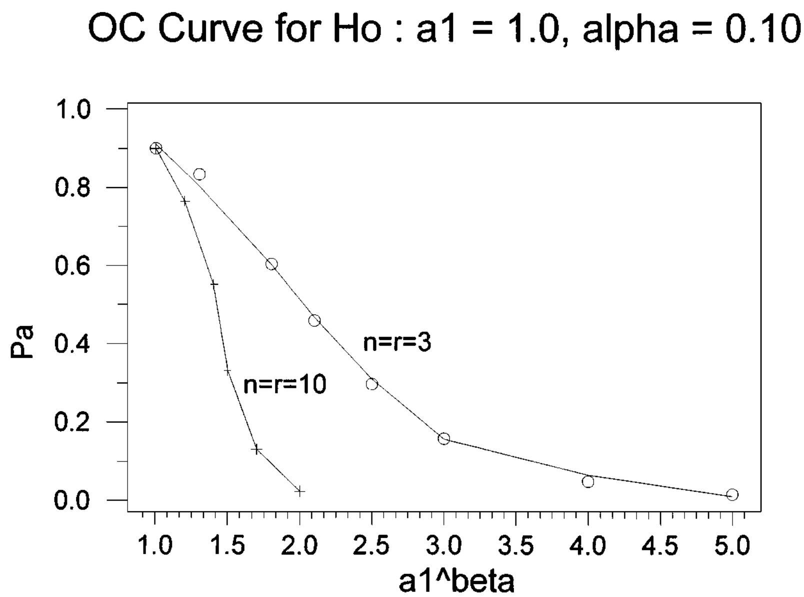


*Figure 1*

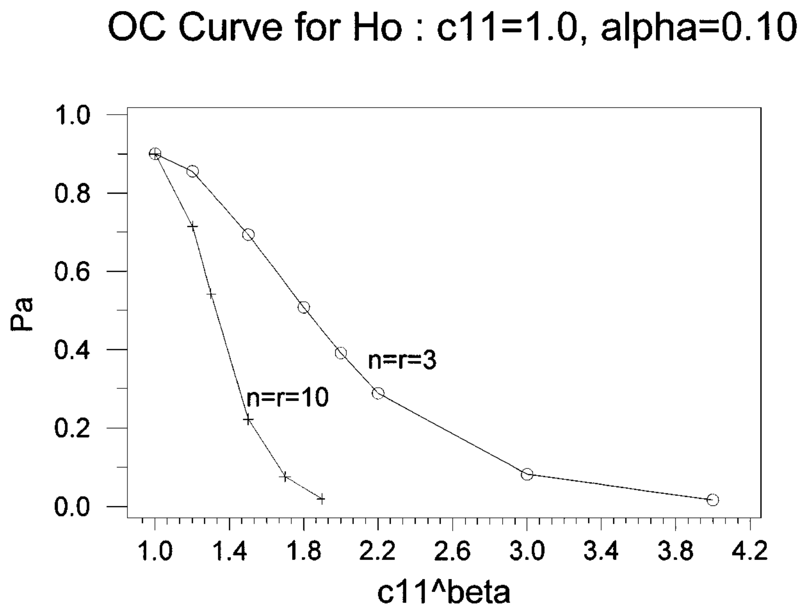


*Figure 2*

having $\eta = 2$ and $\beta = 2$. These data were used to form the first row of the $2 \times 2$ layout. Two further samples of size $n = 5$ were drawn from the Weibull population having $\eta = 1/2$ and $\beta = 2$ to form the second row. The sorted data are tabled below:

| | |
|---|---|
| 0.6297 | 1.021 |
| 0.7960 | 1.107 |
| 0.9468 | 1.502 |
| 2.208 | 1.945 |
| 2.147 | 2.727 |
| 0.1117 | 0.1999 |
| 0.2361 | 0.3451 |
| 0.3038 | 0.6332 |
| 0.3310 | 0.7275 |
| 0.6333 | 0.7447 |

In terms of the model these data represent the case where the base scale parameter $\eta = 1$, there is a row effect $a_1 = 2$ and $a_2 = 1/2$ but no column or interaction effect ($b_1 = b_2 = c_{11} = 1$).

The computed estimates of the shape parameter are listed below:

$$\hat{\beta}_1 = 2.358$$

$$\hat{\beta}_2 = 2.325$$

$$\hat{\beta}_3 = 2.222$$

$$\beta_4 = 1.362$$

$$\beta_5 = 1.354$$

The ratios $\hat{\beta}_1/\hat{\beta}_k$ are tabled below for $k = 2{-}5$, along with the associated $p$ values as estimated from the tabular Monte Carlo distributions of the various ratios as determined under the null hypothesis.

| Effect | Shape parameter ratio | $p$ |
|---|---|---|
| Interaction | $\hat{\beta}_1/\hat{\beta}_2 = 1.014$ | 0.59 |
| Column | $\hat{\beta}_1/\hat{\beta}_3 = 1.061$ | 0.50 |
| Row | $\hat{\beta}_1/\hat{\beta}_4 = 1.731$ | <0.01 |
| All | $\hat{\beta}_1/\hat{\beta}_5 = 1.741$ | <0.01 |

It is seen that the analysis has correctly detected the row effect and, also, correctly failed to show a significant column or interaction effect. The ratio for "ALL" will react to all significant effects. The fact that its magnitude is close to the magnitude of the ratio for the row effect further reflects that only the row effect is significant. The estimated parameters assuming only row effects are meaningful are:

$$\hat{a}_1 = 1/\hat{a}_2 = 1.847, \quad \hat{\eta} = 0.9082, \quad \hat{\beta} = 2.358$$

A 90% confidence interval for $\beta$ may be estimated from Equation 22 using the percentage points of $\hat{\beta}_1/\beta$ listed in Table II.

$$1.40 = 2.358/1.683 < \beta < 2.358/0.859 = 2.75$$

It is noted that this interval includes the true value $\beta = 2.0$. A median unbiased estimate of the shape parameter is computed using Equation 23 and the median value, $v_{0.50}$, of $\hat{\beta}_1/\beta$ listed in Table II as:

$$\hat{\beta}_1'' = 2.358/1.179 = 2.0$$

It is seen that in an unusual coincidence, the median unbiased estimate is exactly equal to the true value of $\beta$.

As a second example of the methodology we will analyze a $2 \times 2$ experiment in which the shear strengths of ten specimens of a polymer material used in dental restorations were measured at all four combinations of the levels of two factors. The material is a 60 : 40 mix of bisphenol A glycidyl methacrylate: triethylene glycol dimethacrylate containing 3 vol % of OX50 colloidal silica (Degussa) used as filler. The factors represented (1) the presence and absence of silanation (a treatment designed to bond the silica filler to the polymer matrix) and (2) whether or not the specimens had been soaked to saturation in a 50 : 50 mixture of ethanol and water. The shear strengths in megapascals (MPa) are listed below for each combination of the factor levels.

| | Soaked | Unsoaked |
|---|---|---|
| | 18.9709 | 36.7600 |
| | 19.5638 | 38.5300 |
| | 20.1566 | 40.9100 |
| | 20.1566 | 41.5000 |
| Silanated | 20.7495 | 42.6800 |
| | 21.3423 | 43.2800 |
| | 22.5280 | 43.2800 |
| | 22.5280 | 43.8700 |
| | 22.5280 | 43.8700 |
| | 23.1208 | 44.4600 |

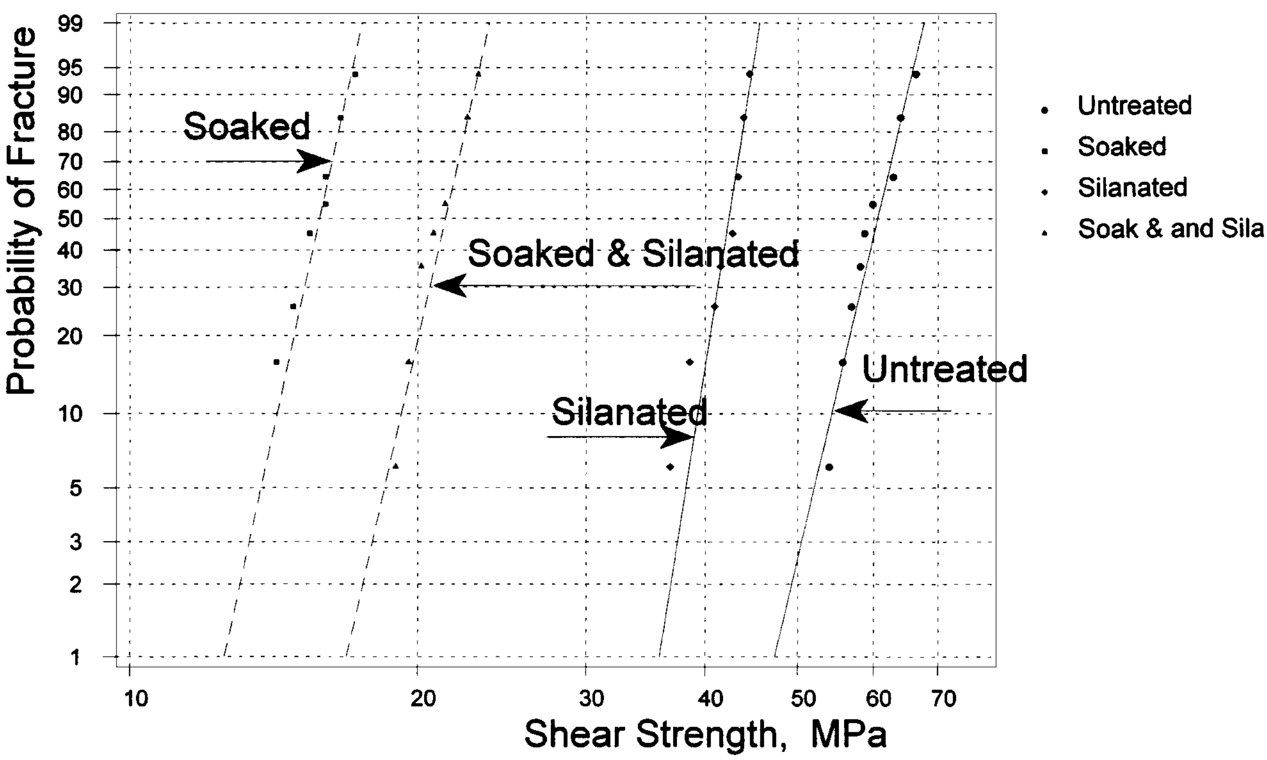


*Figure 3* BISGMA with soaking and silanation.

| | Soaked | Unsoaked |
|---|---|---|
| | 14.2282 | 53.9486 |
| | 14.2282 | 55.7271 |
| | 14.8210 | 56.9128 |
| | 15.4139 | 58.0985 |
| Unsilanated | 15.4139 | 58.6913 |
| | 16.0067 | 59.8770 |
| | 16.0067 | 62.8412 |
| | 16.5996 | 64.0269 |
| | 16.5996 | 64.0269 |
| | 17.192 | 66.3983 |

The validity of the analysis described in this paper depends upon the premise that the data within each cell are drawn from Weibull populations having a common shape parameter. Fig. 3 shows the data within each cell plotted on coordinates for which Weibull distributed data are expected to plot as an approximate straight line. Visually these Weibull "plots" support the assumption that the data follow the two parameter Weibull distributions.

A formal test of the commonality of the shape parameters may be based on the ratio of the largest and smallest among the four ML shape parameter estimates computed from the data within each cell. The individual maximum likelihood shape parameter estimates are tabled below in the positions corresponding to the data table above.

| | |
|---|---|
| $\hat{\beta} = 17.9$ | $\hat{\beta} = 25.4$ |
| $\hat{\beta} = 18.5$ | $\hat{\beta} = 17.1$ |

The ratio of the largest to smallest among these four shape parameter estimates is $\hat{\beta}_{\max}/\hat{\beta}_{\min} = 25.4/17.1 = 1.49$. Under the assumption that $\beta$ is common to all the cells, $\hat{\beta}_{\max}/\hat{\beta}_{\min}$ will follow the distribution discussed in [11] with $n = r = 10$ and $k = 4$. Using tables given in [12] the probability of a ratio larger than 1.49 is found to be about 0.85. There is thus no evidence to suggest that the shape parameter varies from cell-to-cell.

The five estimates of the shape parameter obtained under the five models were computed to be as follows:

$$\hat{\beta}_1 = 19.018$$

$$\hat{\beta}_2 = 6.421$$

$$\hat{\beta}_3 = 2.327$$

$$\hat{\beta}_4 = 6.419$$

$$\hat{\beta}_5 = 2.111$$

The various ratios and the upper 0.01, 0.05, and 0.10 points of their null distribution are tabled below:

| Shape parameter estimates ratio | Upper percentage points 0.10 | 0.05 | 0.01 |
|---|---|---|---|
| $\hat{\beta}_1/\hat{\beta}_2 = 2.96$ | 1.053 | 1.073 | 1.1125 |
| $\hat{\beta}_1/\hat{\beta}_3 = 8.17$ | 1.085 | 1.112 | 1.1171 |
| $\hat{\beta}_1/\hat{\beta}_4 = 2.96$ | 1.085 | 1.112 | 1.1171 |
| $\hat{\beta}_1/\hat{\beta}_5 = 9.013$ | 1.115 | 1.114 | 1.215 |

$\hat{\beta}_1/\hat{\beta}_2$ is significant indicating that the interaction effect is real. The interpretation is that the effect of soaking is greater with unsilanated material than with silanated. That is, silanation acts to diminish the deleterious effect of soaking. When the data exhibit a significant interaction effect the other tests become irrelevant. Each cell of the matrix must be separately estimated subject to the commonality of the shape parameter. The shape parameter used to test for a column effect, $\hat{\beta}_3$, is reduced in magnitude both by the column effect and the interaction effect if there is one. Likewise the row effect shape parameter is reduced in magnitude both by the interaction effect and the actual row effect if any. From the table above the shape parameter ratio for testing the row effect is numerically about the same (2.96) as the ratio for the interaction indicating that the row effect itself is not large. Adopting model 1 as the most appropriate characterization of the data, the estimated effects are calculated to be:

$$\hat{\eta} = 31.086$$
$$\hat{a}_1 = 0.983, \quad \hat{a}_2 = 1.017$$
$$\hat{b}_1 = 0.603, \quad \hat{b}_2 = 1.66$$
$$\hat{c}_{11} = \hat{c}_{22} = 1.18, \quad \hat{c}_{12} = \hat{c}_{21} = 0.847$$

Using these factor estimates the overall shape parameter estimates for each cell may be computed to be:

| | Soaked | Unsoaked |
|---|---|---|
| Silanated | $\hat{\eta}_{11} = 21.84$ | $\hat{\eta}_{12} = 42.8$ |
| Non-silanated | $\hat{\eta}_{21} = 16.1$ | $\hat{\eta}_{22} = 62.1$ |

Using the values in Table II for $n = r = 10$, gives using Equation 22, the following 90% confidence limits for $\beta$.

$$14.06 = \frac{19.02}{1.353} < \beta < \frac{19.02}{0.873} = 21.79$$

Using Equation 23 the median unbiased estimate of the common shape parameter is:

$$\hat{\beta}'' = \frac{19.02}{1.076} = 17.7$$

It is seen that the procedure has determined the Weibull shape parameter to within a narrow range of uncertainty because all 40 data points contribute to its estimation.

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